

Methods for variational inequality problem over the fixed point set of a quasi-nonexpansive operator

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Abstract

Many convex optimization problems in a Hilbert space \mathcal{H} can be written as a variational inequality problem $VIP(\mathcal{F}, C)$ formulated as follows: Given a closed convex subset $C \subset \mathcal{H}$, find $\bar{u} \in C$ such that $\langle \mathcal{F}\bar{u}, z - \bar{u} \rangle \geq 0$ for all $z \in C$, where $\mathcal{F} : \mathcal{H} \rightarrow \mathcal{H}$ is a strongly monotone and Lipschitz continuous operator. We will consider a special case of $VIP(\mathcal{F}, C)$, where $C := \text{Fix}T$ for a quasi-nonexpansive operator $T : \mathcal{H} \rightarrow \mathcal{H}$, i.e., an operator having a fixed point and satisfying $\|Tu - z\| \leq \|u - z\|$ for any $u \in \mathcal{H}$ and $z \in \text{Fix}T$. A standard method for solving $VIP(\mathcal{F}, C)$ is the projected gradient method $u^{k+1} = P_C(u^k - \mu\mathcal{F}u^k)$. For details, see [2, Theorem 46.C]. Unfortunately, the method cannot be applied for solving $VIP(\mathcal{F}, \text{Fix}T)$, because, in general, the metric projection $P_C u$, where $u \in \mathcal{H}$, cannot be evaluated explicitly. In this case one can apply the hybrid steepest descent method (HSDM), $u^{k+1} = Tu^k - \lambda_k \mathcal{F}Tu^k$. For details, see [1, Section 4], where sufficient conditions for the convergence of the method are given. We will consider a generalized HSDM for solving $VIP(\mathcal{F}, \text{Fix}T)$, $u^{k+1} = T_k u^k - \lambda_k \mathcal{F}T_k u^k$, where $T_k : \mathcal{H} \rightarrow \mathcal{H}$, $k \geq 0$, are quasi-nonexpansive operators. We will suppose that $\bigcap_{k=0}^{\infty} \text{Fix}T_k \supseteq \text{Fix}T$ and $\text{Fix}T_k$ approximate $\text{Fix}T$ in some sense. We will give sufficient conditions for the convergence of the generalized HSDM as well as present examples of methods which satisfy these conditions.

Bibliography

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- [2] E. Zeidler, *Nonlinear Functional Analysis and Its Applications, III – Variational Methods and Optimization*, Springer-Verlag, New York, 1985.