# ELEC4410

# Control Systems Design

## *Lecture 15: Observability*

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# Outline

- Observability
- Observability Gramian
- Duality Controllability-Observability
- Observability Tests
- Observation via Differentiation

# Observability

The concept of observability is dual to that of controllability, and deals with the possibility of **estimating** the state of the system from the knowledge of its inputs and outputs.

Consider the LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times q} \\ y &= Cx + Du, \quad C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times q} \end{aligned} \tag{SE}$$

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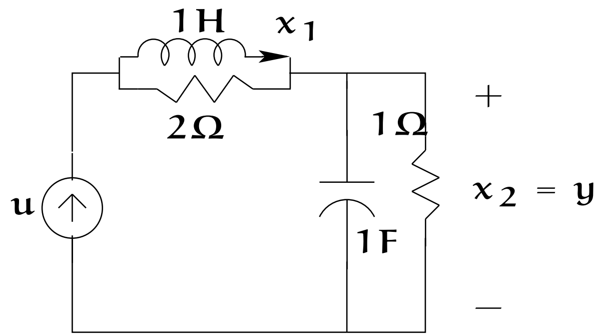
**Observability:** The state equation (SE), or the pair $(A, C)$, is said to be observable if for any unknown initial state $x(0)$, there exists a finite time $t_1 > 0$ such that the knowledge of the input $u(t)$ and the output $y(t)$ over $[0, t_1]$ suffices to determine uniquely the initial state $x(0)$. Otherwise, the equation is said to be unobservable.

# *Observability*

**Example (Unobservable systems).** The network shown in the figure below has two state variables: the current $x_1$ through the inductor and the voltage $x_2$ across the capacitor. The input $u$ is a current source.




If $u = 0$, $x_2(0) = 0$ and $x_1(0) = a \neq 0$, then the output is identically zero. Any $x(0) = \left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right]$ and $u(t) \equiv 0$ yield the same output $y(t) \equiv 0$.

Thus there is no way to uniquely determine the initial state $\left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right]$ and the system is unobservable. □

# Observability

We have shown that the response of the state equation system is given by

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}BU(\tau)d\tau + Du(t)$$

In studying observability we **assume $u$ and $y$ known; the initial state $x(0)$ is the only unknown**.

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In studying observability we **assume $u$ and $y$ known; the initial state $x(0)$ is the only unknown**. From the previous equation,

$$\mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) = \bar{\mathbf{y}}(t), \tag{1}$$

where

$$\bar{\mathbf{y}}(t) = \mathbf{y}(t) - \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau - \mathbf{D}\mathbf{u}(t)$$

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$$\bar{y}(t) = y(t) - C\int_0^t e^{A(t-\tau)}BU(\tau)d\tau - Du(t)$$

is a known function. Thus the observability problem reduces to **finding $x(0)$ as the unique solution of (1)**.

# Observability

For a fixed time $t$, $Ce^{At}$ is a $p \times n$ real, constant matrix, and $\bar{y}(t)$ a constant $p \times 1$ vector.

Thus, in general, because $p < n$ (there are less outputs than states) we cannot find a **unique** vector $x(0)$ from

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$$\boxed{Ce^{At}x(0) = \bar{y}(t)}, \quad \text{for a given fixed } t.$$

To determine $x(0)$ uniquely we need to use the knowledge of $y(t)$ and $u(t)$ over **a nonzero time interval**.

# Observability Gramian

# *Observability Gramian*

**Theorem (Gramian Observability Test).** The state equation (SE) is observable if and only if the $n \times n$ matrix

$$W_o(t) = \int_0^t e^{A^T \tau} C^T C e^{A\tau} d\tau \qquad \text{(WO)}$$

is nonsingular for any $t > 0$.

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**Note** that observability only depends on the matrices $A$ and $C$.

# Observability Gramian

If the matrix $A$ is **Hurwitz** (all eigenvalues have negative real part), then $W_o(t)$ converges for $t \to \infty$, and we simply denote it by $W_o$,

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In MATLAB the functions `Ob = obsv(A,C)` and `Wo = gram(A',C')'` respectively compute the observability matrix $\mathcal{O}$ and Gramian $W_o$. By checking the rank of $\mathcal{O}$ or $W_o$, we can determine if a pair $(A, C)$ is observable.

# Duality Controllability-Observability

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**Theorem (Duality).** The pair $(\mathbf{A}, \mathbf{B})$ is controllable if and only if the pair $(\mathbf{A}^{\mathrm{T}}, \mathbf{C}^{\mathrm{T}})$ is observable.

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*Proof.* The pair $(\mathbf{A}, \mathbf{B})$ is controllable if and only if

$$\mathbf{W_c}(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau$$

is nonsingular for any $t$.

On the other hand, the pair $(\mathbf{A}^T, \mathbf{B}^T)$ is observable if and only if, by replacing $\mathbf{A}$ by $\mathbf{A}^T$ and $\mathbf{C}$ by $\mathbf{B}^T$ in (WO),

$$\mathbf{W_o}(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau$$

is nonsingular for any $t$; the two conditions are thus identical. □

# *Duality Controllability-Observability*

The Duality between controllability and observability establishes that we can test the observability of a pair $(\mathbf{A}, \mathbf{C})$ by using the controllability tests that we already know on the pair $(\mathbf{A}^{\mathsf{T}}, \mathbf{C}^{\mathsf{T}})$.

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**Example.** Consider the system

$$\dot{x}(t) = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x(t)$$

By duality, we can check the observability of this system as the controllability of $(A^T, C^T)$; for instance, via the matrix

$$\mathcal{C} = \begin{bmatrix} C^T & A^T C^T & A^{T\,2} C^T \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & -5 & 16 \\ 1 & -4 & 11 \end{bmatrix}$$

which has rank $3$, thus $(A, C)$ is observable. $\square$

# Test for Observability

# Observability Tests

**Theorem (Observability Tests).** The following statements are equivalent.

1. The $n$-dimensional pair $(A, C)$, $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$, is observable.

2. The **Observability Matrix**

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}, \qquad \mathcal{O} \in \mathbb{R}^{np\times n},$$

   has rank $n$ (full column rank).

3. The $n \times n$ matrix $W_o(t) = \int_0^t e^{A\tau} B B^T e^{A^T\tau} d\tau$ is non singular for all $t > 0$.

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$$\begin{bmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{bmatrix} x(0) = \begin{bmatrix} \bar{y}(0) \\ \dot{\bar{y}}(0) \\ \cdots \\ \bar{y}^{n-1}(0) \end{bmatrix} \quad \text{i.e.,} \quad \boxed{\mathcal{O}x(0) = \vec{y}(0)}.$$

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If the system is observable, then $\mathcal{O}$ is full column rank, and we know there exist a unique solution of $\mathcal{O}x(0) = \vec{y}(0)$ given by

$$x(0) = \left[\mathcal{O}^T\mathcal{O}\right]^{-1}\mathcal{O}^T\vec{y}(0).$$

Note that we still need to know $\bar{y}(t)$ on a neighbourhood of $t = 0$ to be able to determine $\vec{y}(0)$.

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Although theoretically we could obtain $x(0)$ by differentiation, in practice it is **not** recommended, since

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- on the other hand, **integration** "averages" high frequency noise, diminishing its effects in the computation of $x(0)$.

It is much more convenient to implement observation by using integration, e.g., via the formula

$$x(0) = W_o^{-1}(t_1)\int_0^{t_1} e^{A^T\tau}C^T\bar{y}(\tau)d\tau.$$

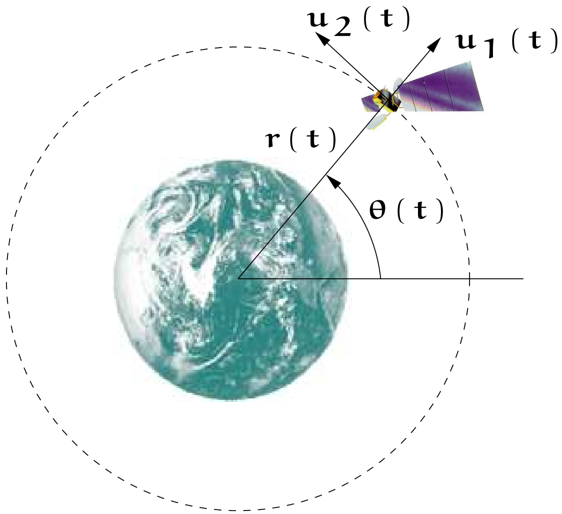
# Examples

# Observability Examples

**Example (Earth satellite).**
A linearised state equation for a satellite in circular orbit is given by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2r_0\omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2\omega_0}{r_0} & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r_0} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t) = \begin{bmatrix} r(t) \\ \theta(t) \end{bmatrix}$$




where the first output is the (incremental) radial distance $r(t)$ and the second the (incremental) angle $\theta(t)$.

The position of the satellite can be adjusted by means of the thrust forces $u_1(t)$ and $u_2(t)$. The nominal radius is $r_0$ and the nominal angular velocity $\omega_0$.

# Observability Examples

## Example (continuation).

- Suppose that only **radial** distance measurements

$$y_1(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t) = C_1 x(t)$$

are available on a specified time interval. The observability matrix in this case is

$$\begin{bmatrix} C_1 \\ AC_1 \\ A^2C_1 \\ A^3C_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3\omega_0^3 & 0 & 0 & 2r_0\omega_0 \\ 0 & -\omega_0^2 & 0 & 0 \end{bmatrix}$$ which has rank $3$.

Therefore, radial measurement does **not** suffice to compute the complete orbit state.

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- On the other hand, measurement of angle,

$$y_1(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x(t) = C_2 x(t)$$

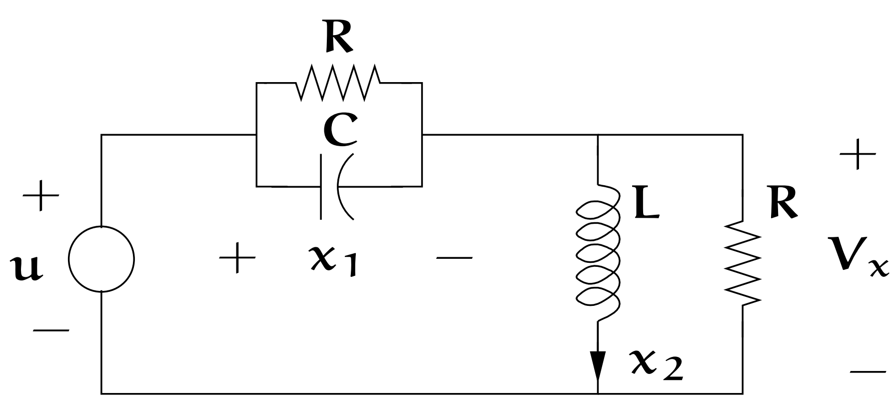
does suffice, as can be readily verified. □

# Observability Examples

**Example (Controllability and Observability of an RLC circuit).** The RLC circuit below is modelled by the state equations

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{2}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} \\ \frac{1}{L} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + u(t)$$



# Observability Examples

**Example (RLC circuit continuation).** We test controllability by checking the rank of the Controllability Matrix,

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \frac{1}{RC} & -\frac{2}{R^2C^2} + \frac{1}{LC} \\ \frac{1}{L} & -\frac{1}{RLC}. \end{bmatrix}$$

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The determinant is zero (and thus the system **uncontrollable**) if

$$\frac{1}{R^2LC^2} - \frac{1}{L^2C} = 0 \Leftrightarrow \boxed{R = \sqrt{\frac{L}{C}}}$$

# *Observability Examples*

**Example (RLC circuit continuation).** On the other hand, the Observability Matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ \frac{2}{RC} & -\frac{1}{C} \end{bmatrix}$$

which is obviously full rank.

Hence the system is **always** observable, but becomes uncontrollable whenever $R = \sqrt{L/C}$.

# Observability Examples

**Example (RLC circuit continuation).** Let's see what happens to the system transfer function when controllability is lost.

The calculation, using the known formula $\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ gives

$$\mathbf{G}(s) = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} s + \frac{2}{RC} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{RC} \\ \frac{1}{L} \end{bmatrix} + 1$$

$$= \frac{s\left(s + \frac{1}{RC}\right)}{s^2 + \frac{2}{RC}s + \frac{1}{LC}}$$

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**Example (RLC circuit continuation).** The poles of the circuit transfer function are

$$s_{1,2} = -\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{1}{LC}}.$$

Both roots have negative real part, and thus conclude that the system is **asymptotically stable** and **BIBO stable** for any value of $R$, $L$ and $C$.

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In particular, for $R = \sqrt{L/C}$ (the value for which the system becomes uncontrollable), we have

$$s_{1,2} = -\frac{1}{RC} \pm \sqrt{\frac{1}{LC} - \frac{1}{LC}} = -\frac{1}{RC},$$

that is, the system has repeated roots, and

$$G(s) = \frac{s\left(s + \frac{1}{RC}\right)}{\left(s + \frac{1}{RC}\right)^2} = \frac{s}{\left(s + \frac{1}{RC}\right)}.$$

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A pole-zero cancellation reduces the system to first order. □

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# *Summary*

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- If a system is observable, it is possible to find $x(0)$ from measurements of $u(t)$ and $y(t)$ over a time interval $t \in [0, t_1], t_1 > 0$.
- Observability depends o the matrices $A$ and $C$ of the state equation of the system. The pair $(A, C)$ is observable if and only if

$$\textbf{rank}\ \mathcal{O} = \textbf{rank} \begin{bmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{bmatrix} = n$$

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- As for Controllability, Observability is invariant with respect to change of coordinates (algebraic equivalence transformations).