ELEC4410

Control Systems Design

Lecture 16: Controllability and Observability Canonical Decompositions

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Outline

- Canonical Decompositions
- Kalman Decomposition and Minimal Realisation
- Discrete-Time Systems

Canonical Decompositions

The **Canonical Decompositions** of state equations will establish the relationship between Controllability, Observability , and a transfer matrix and its minimal realisations.

Consider the state equation

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \text{where} \quad \begin{aligned} &A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}, \\ &C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{q\times p}. \end{aligned} \qquad \text{(SE)}$$

Let $\bar{x} = Px$, where P is nonsingular, $P \in \mathbb{R}^{n\times n}$. Then we know that the state equation

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} + \bar{D}u \end{aligned} \quad \text{where} \quad \begin{aligned} &\bar{A} = PAP^{-1}, \bar{B} = PB, \\ &\bar{C} = CP^{-1}, \bar{D} = D, \end{aligned}$$

is **algebraically equivalent** to (SE).

Canonical Decompositions

Theorem (Controllable/Uncontrollable Decomposition). Consider the n-dimensional state equation (SE) and suppose that

$$\text{rank}\,\mathcal{C} = \text{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n_1 < n$$

(i.e., the system is not controllable). Let the $n \times n$ matrix of change of coordinates P be defined as

$$P^{-1} = \begin{bmatrix} q_1 & q_2 & \cdots & q_{n_1} & \cdots & q_n \end{bmatrix}$$

where the first n_1 columns are any n_1 *independent* columns in $\mathcal{C}$, and the remaining are arbitrarily chosen so that P is nonsingular. Then the equivalence transformation $\bar{x} = Px$ transforms (SE) to

$$\begin{bmatrix} \dot{\bar{x}}_{\mathcal{C}} \\ \dot{\bar{x}}_{\tilde{\mathcal{C}}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{\mathcal{C}} & \bar{A}_{12} \\ 0 & \bar{A}_{\tilde{\mathcal{C}}} \end{bmatrix} \begin{bmatrix} \bar{x}_{\mathcal{C}} \\ \bar{x}_{\tilde{\mathcal{C}}} \end{bmatrix} + \begin{bmatrix} \bar{B}_{\mathcal{C}} \\ 0 \end{bmatrix} u$$

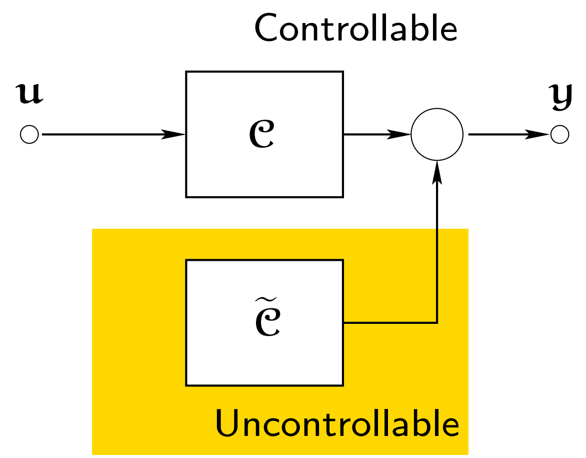
$$y = \begin{bmatrix} \bar{C}_{\mathcal{C}} & \bar{C}_{\tilde{\mathcal{C}}} \end{bmatrix} \bar{x} + Du$$

Canonical Decompositions

The states in the new coordinates are decomposed into

$\bar{x}_{\mathcal{C}}$: n_1 controllable states

$\bar{x}_{\widetilde{\mathcal{C}}}$: $n - n_1$ uncontrollable states



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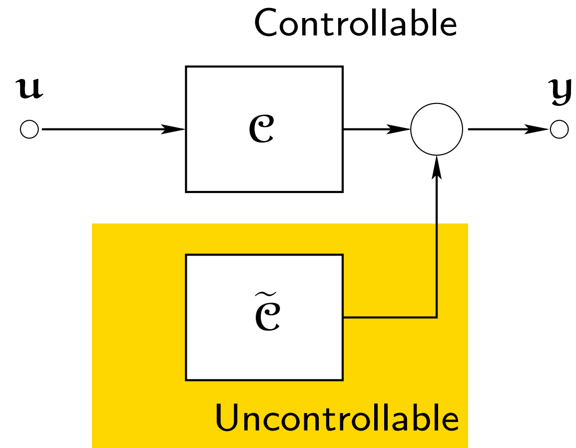


The **reduced order** state equation of the controllable states

$$\dot{\bar{x}}_{\mathcal{C}} = \bar{A}_{\mathcal{C}}\bar{x}_{\mathcal{C}} + \bar{B}_{\mathcal{C}}u$$

$$\bar{y} = \bar{C}_{\mathcal{C}}\bar{x} + Du$$

is controllable and has the same transfer function as the original state equation (SE).

The MATLAB function `ctrbf` transforms a state equation into its controllable/uncontrollable canonical form.

Canonical Decompositions

Theorem (Observable/Unobservable Decomposition). Consider the n-dimensional state equation (SE) and suppose that

$$\textbf{rank}\,\mathcal{O} = \textbf{rank}\begin{bmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{bmatrix} = n_2 < n \quad \text{(i.e., the system is not observable).}$$

Let the $n \times n$ matrix of change of coordinates P be defined as

$$P = \begin{bmatrix} p_1 \\ p_2 \\ \cdots \\ p_{n_2} \\ \cdots \\ p_n \end{bmatrix}$$

where the first n_2 columns are any n_2 *independent* columns in $\mathcal{O}$, and the remaining are arbitrarily chosen so that P is nonsingular. Then the equivalence transformation $\bar{x} = Px$ transforms (SE) to

$$\begin{bmatrix} \dot{\bar{x}}_{\mathcal{O}} \\ \dot{\bar{x}}_{\tilde{\mathcal{O}}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{\mathcal{O}} & 0 \\ \bar{A}_{21} & \bar{A}_{\tilde{\mathcal{O}}} \end{bmatrix} \begin{bmatrix} \bar{x}_{\mathcal{O}} \\ \bar{x}_{\tilde{\mathcal{O}}} \end{bmatrix} + \begin{bmatrix} \bar{B}_{\mathcal{O}} \\ \bar{B}_{\tilde{\mathcal{O}}} \end{bmatrix} u$$

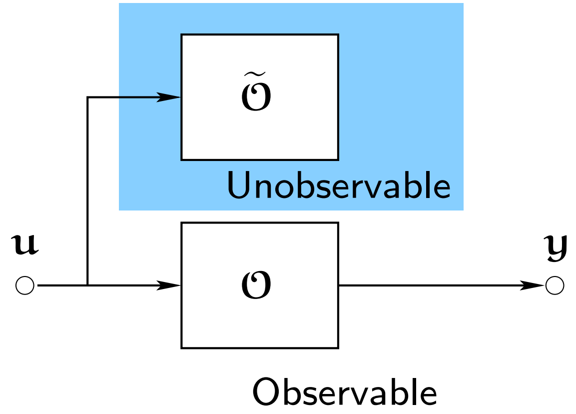
$$y = \begin{bmatrix} \bar{C}_{\mathcal{O}} & 0 \end{bmatrix} \bar{x} + Du$$

Canonical Decompositions

The states in the new coordinates are decomposed into

$\bar{\mathbf{x}}_{\mathfrak{O}}$: n_2 observable states

$\bar{\mathbf{x}}_{\tilde{\mathfrak{O}}}$: $n - n_2$ unobservable states



Canonical Decompositions

The states in the new coordinates are decomposed into

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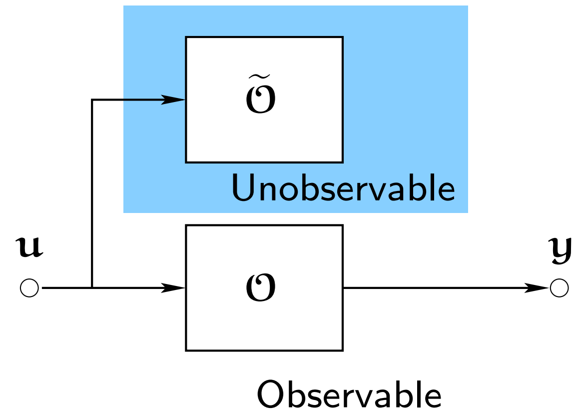


The **reduced order** state equation of the observable states

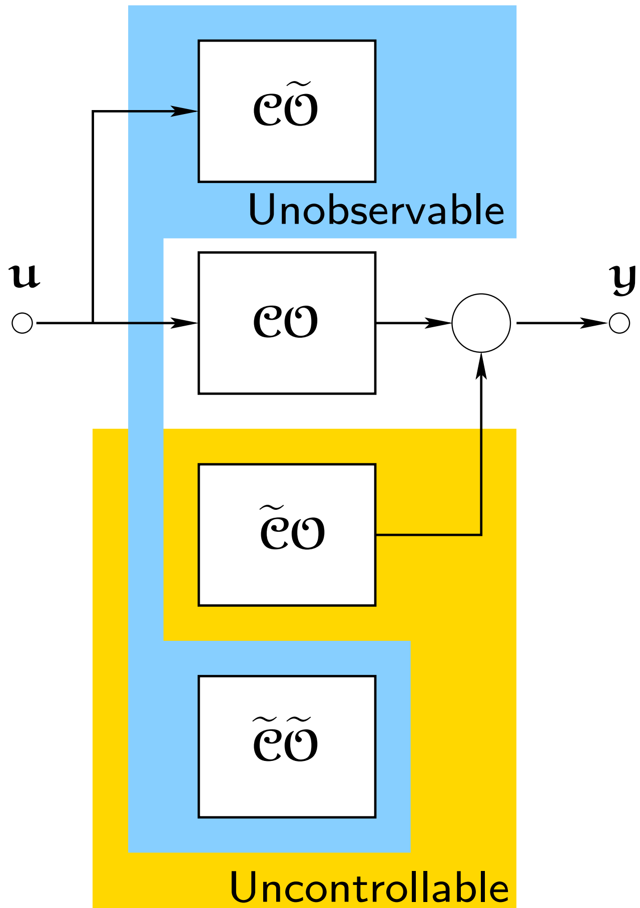
$$\dot{\bar{x}}_{\mathcal{O}} = \bar{A}_{\mathcal{O}} \bar{x}_{\mathcal{O}} + \bar{B}_{\mathcal{O}} u$$
$$\bar{y} = \bar{C}_{\mathcal{O}} \bar{x} + D u$$

is observable and has the same transfer function as the original state equation (SE). □

The MATLAB function `obsvf` transforms a state equation into its observable/unobservable canonical form.

Kalman Decomposition

The **Kalman decomposition** combines the Controllable/Uncontrollable and Observable/Unobservable decompositions.



Every state-space equation can be transformed, by equivalence transformation, into a canonical form that splits the states into

- Controllable and observable states

Kalman Decomposition

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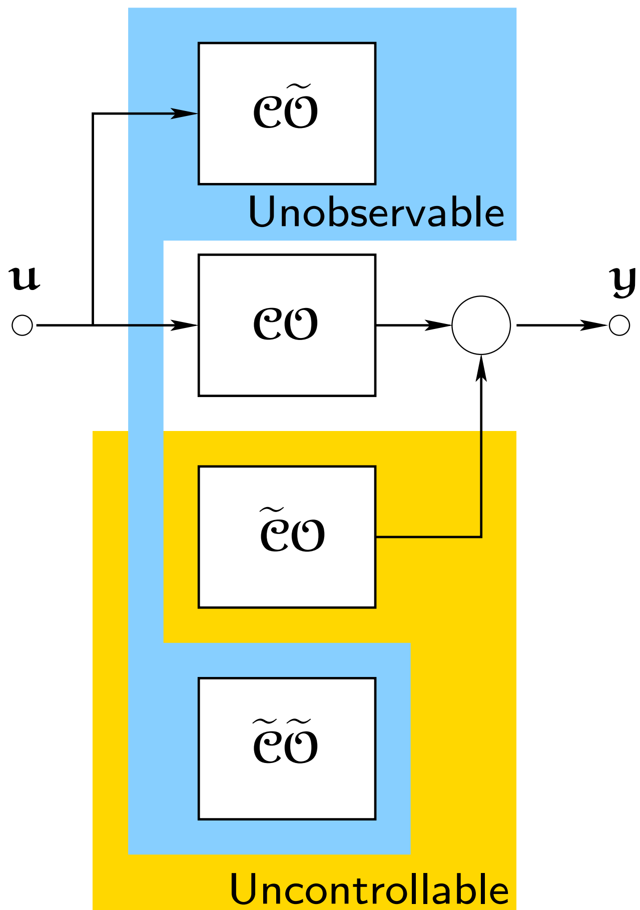

Every state-space equation can be transformed, by equivalence transformation, into a canonical form that splits the states into

- Controllable and observable states
- Controllable but unobservable states

Kalman Decomposition

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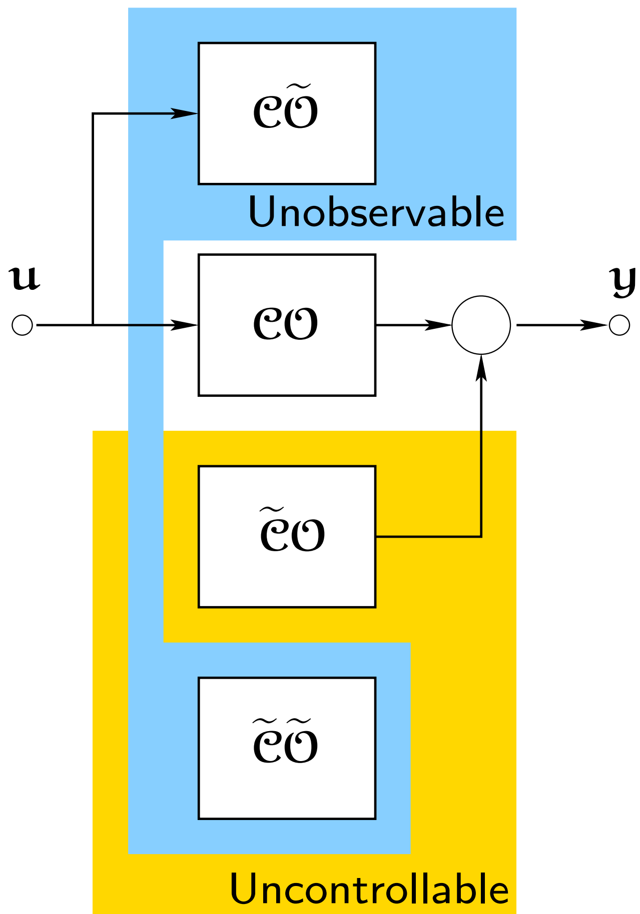

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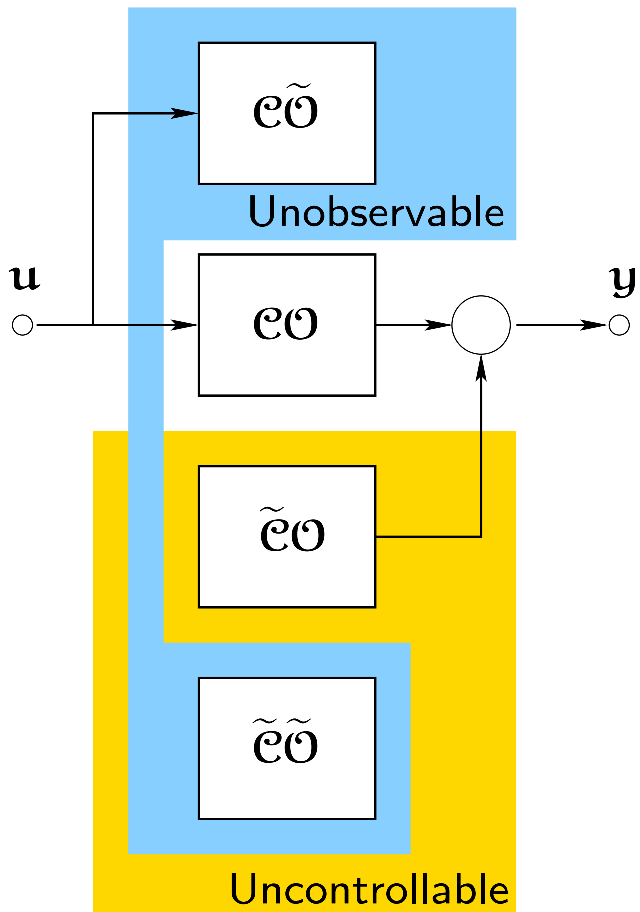

Every state-space equation can be transformed, by equivalence transformation, into a canonical form that splits the states into

- Controllable and observable states
- Controllable but unobservable states
- Uncontrollable but observable states
- Uncontrollable and unobservable states

Kalman Decomposition

The **Kalman decomposition** brings the system to the form

$$\begin{bmatrix} \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{c\tilde{o}} \\ \dot{\bar{x}}_{\tilde{c}o} \\ \dot{\bar{x}}_{\tilde{c}\tilde{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\tilde{o}} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\tilde{c}o} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\tilde{c}\tilde{o}} \end{bmatrix} \underbrace{\begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\tilde{o}} \\ \bar{x}_{\tilde{c}o} \\ \bar{x}_{\tilde{c}\tilde{o}} \end{bmatrix}}_{\bar{x}} + \begin{bmatrix} \bar{B}_{co} \\ \bar{B}_{c\tilde{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_{co} & 0 & \bar{C}_{\tilde{c}o} & 0 \end{bmatrix} \bar{x} + Du$$

Kalman Decomposition

The **Kalman decomposition** brings the system to the form

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$$y = \begin{bmatrix} \bar{C}_{co} & 0 & \bar{C}_{\tilde{c}o} & 0 \end{bmatrix} \bar{x} + Du$$

A **minimal realisation** of the system is obtained by using only the **controllable and observable states** from the Kalman decomposition.

$$\dot{\bar{x}}_{co} = \bar{A}_{co}\bar{x}_{co} + \bar{B}_{co}u$$

$$\bar{y} = \bar{C}_{co}\bar{x} + Du$$

Kalman Decomposition

Example. Consider the system in Modal Canonical Form

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix} x$$

From the example seen in the Tutorial, *Controllability and Observability in Modal Form equations*, we see that

- the first λ_1 is controllable and observable
- λ_2 is not controllable, although observable
- λ_3 is controllable and observable

Thus a minimal realisation of this system is given by

$$\dot{\bar{x}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \bar{x}$$

with transfer function $G(s) = \frac{1}{s - \lambda_1} + \frac{2}{s - \lambda_3}$

Canonical Decompositions

Example (Controllable/Uncontrollable decomposition). Consider the third order system

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x\end{aligned}$$

Compute the rank of the controllability matrix,

$$\textbf{rank}\, \mathcal{C} = \textbf{rank} \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \textbf{rank} \begin{bmatrix} 0 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \end{bmatrix} = 2 < 3,$$

thus the system is **not** controllable. Take the change of coordinates formed by the first two columns of $\mathcal{C}$ and an arbitrary third one independent of the first two,

$$P^{-1} = Q \triangleq \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Canonical Decompositions

Example (continuation). By doing $\hat{x} = \mathbf{P}x$ we obtain the equivalent equations

$$\dot{\hat{x}} = \begin{bmatrix} 1 & 0 & \vdots & 0 \\ 1 & 1 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \cdots & \cdots \\ 0 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 & \vdots & 1 \end{bmatrix} x$$

and the reduced controllable system

$$\dot{\hat{x}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} x$$

which has the same transfer matrix than the original system

$$\mathbf{G}(s) = \begin{bmatrix} \frac{s+1}{s^2-2s+1} & \frac{2}{s-1} \end{bmatrix}.$$

Discrete-Time Systems

For controllability and observability of a discrete-time equation

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}$$

we can use the same **Controllability** and **Observability** matrices rank tests that we have for continuous-time systems,

$$\textbf{rank}\ \mathcal{C} = \textbf{rank}\left[\, B \ \ AB \ \ \dots \ \ A^{n-1}B \,\right] = n \quad \Leftrightarrow \quad \text{Controllability}$$

$$\textbf{rank}\ \mathcal{O} = \textbf{rank}\begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix} = n \quad \Leftrightarrow \quad \text{Observability}$$

Canonical decompositions are analogous.

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- The **controllable** and **observable** part of a state equation yields minimal realisation.

Thus, we conclude that for a state equation

minimal realisation $\Leftrightarrow$ controllable and observable